THEORY OF FLOWS OF A TWO-VELOCITY CONTINUOUS MEDIUM CONTAINING SOLID OR LIQUID PARTICLES

(K TEORII TECHENII DVUSKOROSTNOI SPLOSHNOI SREDY S TVERDYMI ILI ZHIDKIMI CHASTITSAMI)

PMM Vol. 29, No. 3, 1965, pp. 418-429

A.N. KRAIKO and L.E. STERNIN
(Moscow)

(Received December 13, 1964)

The problem of the motion of a continuous medium containing heterogeneous particles is an exceedingly complex one. Provided certain conditions are fulfilled, however, the phenomenon can be described with a sufficient degree of accuracy with the aid of the two-velocity continuous medium model. These conditions are as follows: (1) the particles are spheres of uniform size, collisions among which may be neglected; (2) the distances at which the flow characteristics undergo substantial change are much larger than the interparticle distances outside the discontinuity surfaces; (3) the Mach number of the relative motion of the particles is subcritical. In addition, it is assumed that viscosity and thermal conduction are of importance only in gas-particle interaction processes.

Flows of this type are dealt with in a large number of papers, although most investigators have confined their discussions to the one-dimensional stationary case. The present authors are familiar with just three studies of somewhat broader scope. Kh. A. Rakhmatulin [1] obtained the equations of continuity and motion and examined one-dimensional nonstationary flows with plane waves; Ia. Z. Kleiman [2] studied and classified second-order discontinuities. In order to close the system of equations, both authors made use of relations equivalent to the assumption of barotropy. Such an assumption cannot be made for most media, including gases. Finally, Kliegel and Nickerson [3] developed equations for the characteristics of stationary axisymmetrical flow. They did not assume barotropy, but did however, neglect the volume occupied by the particles.

The treatment of the problem in this paper does not rest on any such assumptions. In the general case the equations of continuity, motion, and energy are obtained in integral and differential forms. Discontinuity surfaces are considered. One-dimensional nonstationary and two-dimensional stationary flows are investigated in some detail.

1. Let us consider the motion of a continuous medium containing solid or liquid spherical particles in the case where the size of the particles is small compared with the distance characteristic of substantial changes in the flow parameters. Let m be the mass, ρ_d^0 the density, me_d the internal energy, $\mathbf{V}_d$ the velocity, and T_d the temperature of a

particle, p the pressure, T the gas temperature, $\mathbf{V}$ the velocity of the gas, and t the time. We assume that the force exerted on a particle by the gas is the sum of a force proportional to the pressure gradient and the force $m\mathbf{f}$ due to the viscosity of the medium. In determining the first force we assume that the particle velocity is equal to the velocity of the gas at a given point; in determining the second force we assume that homogeneous flow about the particle proceeds with a velocity $\mathbf{V} - \mathbf{V}_d$. We likewise determine the heat flux mq from the gas to the particle due to the thermal conductivity of the medium. The particle can also receive heat mQ_d and be acted upon by a force $m\mathbf{F}_d$ (through the action of external sources).

Since condition (2) permits us to associate certain values of $\mathbf{V}_d$ and e_d with each point in the flow, the equations of particle motion and energy are

$$(\mathbf{V}_d\nabla)\,\mathbf{V}_d + \frac{\partial \mathbf{V}_d}{\partial t} + \frac{1}{\rho_d^\circ}\nabla p - \mathbf{f} - \mathbf{F}_d = 0 \tag{1.1}$$

$$\mathbf{V}_d\nabla e_d + \frac{\partial e_d}{\partial t} - q - Q_d = 0^* \tag{1.2}$$

$$\mathbf{f} = \varphi^1\cdot|\mathbf{V} - \mathbf{V}_d|^n\,(\mathbf{V} - \mathbf{V}_d), \qquad q = \varphi^2\cdot(T - T_d)^k$$
$$T_d = T_d\,(e_d), \quad \varphi^i = \varphi^i\,(p, T, T_d, |\mathbf{V} - \mathbf{V}_d|), \quad n > -1, \quad k > 0 \tag{1.3}$$

The right-hand sides in (1.3) are known either from the solution of the problem of the homogeneous flow of a viscous heat-conducting gas about a sphere or from experiment.

2. To construct the model of a two-velocity continuous medium, along with the true gas and particle densities ρ^0 and ρ_d^0, we also introduce the densities $\rho = \Delta M/\Delta\tau$ and $\rho_d = \Delta M_d/\Delta\tau$, where ΔM and ΔM_d are the masses of the gas and particles in a physically infinitesimal volume $\Delta\tau$. Thus,

$$\rho = \rho^\circ\,(1 - \rho_d\,/\,\rho_d^\circ) \tag{2.1}$$

The definition of ρ_d is meaningful in one of two cases: either when $\Delta\tau$ contains a sufficiently large number of particles (condition (2) is satisfied) or when the number of particles is very small and we can set $\rho_d = 0$. The other characteristics of the gas and particles are introduced along with ρ and ρ_d, so that the medium containing heterogeneous particles is replaced by two interacting continuous media: the gas as such and the 'gas' of particles; the properties of the latter are assumed to vary in accordance with (1.1)-(1.3). The results of this analysis remain valid even if one takes into account the effect of other particles on the flow around each individual particle. Should this effect indeed be considered, ϕ^i in (1.3) also depend on ρ_d, since $\mathbf{f}$ and q are here determined from the solution of the problem of gas flow in a symmetrical particle lattice.

The continuity equations of both media are derived in the usual way and may be written in integral form as

$$\iiint_{\tau} \frac{\partial \rho}{\partial t} d\tau + \iint_{S} \rho \mathbf{V} \mathbf{n} dS = 0, \qquad \iiint_{\tau} \frac{\partial \rho_d}{\partial t} d\tau + \iint_{S} \rho_d \mathbf{V}_d \mathbf{n} dS = 0 \tag{2.2}$$

where τ is an arbitrary volume bounded by the surface S and $\mathbf{n}$ is the outer normal to S.

To obtain the equations of motion and energy, let us consider the system of gas and particles enclosed by the arbitrary surface S bound to the fixed gas particles. The change in momentum of such a system over a time dt is due to the force $\mathbf{F}_d$ acting on the particles, the external mass force $\mathbf{F}$ acting on the gas, and the force acting on the boundary of the system. In addition, we must take into account the momentum flux associated with the transfer of particles through the surface S with velocity $\mathbf{V}_d - \mathbf{V}$. We note that the surface force is the resultant of pressure forces only, since the viscosity in the gas is assumed to be zero, and the force exerted by the particles on the gas in a volume $d\tau$ is $\mathbf{f}\rho_d d\tau$ and therefore three-dimensional. Taking these considerations into account and carrying out the usual transformations [4] we obtain the equation of motion in integral form,

$$\iiint_{\tau} \rho \left[(\mathbf{V}\nabla)\left(\mathbf{V} + \frac{\rho_d}{\rho}\mathbf{V}_d\right) + \frac{\partial}{\partial t}\left(\mathbf{V} + \frac{\rho_d}{\rho}\mathbf{V}_d\right) - \mathbf{F} - \frac{\rho_d}{\rho}\mathbf{F}_d\right] d\tau + \iint_{S} \{p\mathbf{n} + \rho_d \mathbf{V}_d [(\mathbf{V}_d - \mathbf{V})\mathbf{n}]\} dS = 0 \tag{2.3}$$

In order to find the work done by the surface forces in deriving the energy equation, we must take into account the difference in velocities of portions of the surface element dS occupied by the gas and particles. As shown in [1], the portion of an element dS occupied by the gas is $(\rho/\rho^\circ)\, dS$; the area occupied by the particles is $(\rho_d/\rho_d^\circ)dS$. In view of this fact and the flow of energy carried by the particles over the boundary S, we can carry out the same transformations as in our derivation of (2.3) to obtain

$$\iiint_{\tau} \rho \left\{ \mathbf{V}\nabla \left[\frac{V^2}{2} + e + \frac{\rho_d}{\rho}\left(\frac{V_d^2}{2} + e_d\right)\right] + \frac{\partial}{\partial t}\left[\frac{V^2}{2} + e + \frac{\rho_d}{\rho}\left(\frac{V_d^2}{2} + e_d\right)\right] - Q - \mathbf{F}\mathbf{V} - \frac{\rho_d}{\rho}(Q_d + \mathbf{F}_d\mathbf{V}_d)\right\} d\tau + \iint_{S} \left[p\left(\frac{\rho}{\rho^\circ}\mathbf{V} + \frac{\rho_d}{\rho_d^\circ}\mathbf{V}_d\right) + \rho_d(\mathbf{V}_d - \mathbf{V})\left(\frac{V_d^2}{2} + e_d\right)\right] \mathbf{n} dS = 0 \tag{2.4}$$

where $V = |\mathbf{V}|$, e is the internal energy of the gas and $\rho Q d\tau$ is the heat received from external sources by the gas within a volume $d\tau$.

Equations (1.1)-(1.3) and (2.1)-(2.4) describe the flows of a two-velocity medium. The system is closed by the expressions for ρ° and e (or the specific enthalpy h of the gas) in terms of p and T,

$$\rho^{\bullet} = \rho^{\circ}(p, T), \qquad e = e(p, T), \qquad h \equiv e + p / \rho^{\circ} = h(p, T) \tag{2.5}$$

It will be convenient for us to assume that the variables have been reduced to dimensionless form. This may be achieved by referring the spatial coordinates to the characteristic dimension of the problem l, the velocities to V_{∞}, the time to l/V_{∞}, the densities to ρ_{∞}, the pressure to $\rho_{\infty}V_{\infty}^2$, the internal energies to V_{∞}^2, the forces to V_{∞}^2/l, the thermal fluxes to V_{∞}^3/l, and the temperatures to V_{∞}^2/R, where V_{∞} and ρ_{∞} are the characteristic dimensional velocity and R is the gas constant for the gas. After reduction to dimensionless form, dimensionless parameters appear on the right-hand sides of (1.3) and (2.5). These parameters enter as factors into ϕ^i and characterize the degree of gas-particle interaction.

3. The integral equations can be replaced by differential equations in the flow regions not containing second-order discontinuities. Converting from surface to volume integrals, taking into account the arbitrariness of the volume τ, and performing certain manipulations with the aid of (1.1) and (1.2), we arrive at the equations of continuity, motion, and energy in differential form,

$$\begin{gathered}
\nabla(\rho \mathbf{V}) + \frac{\partial \rho}{\partial t} = 0, \qquad \nabla(\rho_d \mathbf{V}_d) + \frac{\partial \rho_d}{\partial t} = 0 \\
(\mathbf{V}\nabla)\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{\rho^{\circ}}\nabla p + \frac{\rho_d}{\rho}\mathbf{f} - \mathbf{F} = 0 \\
\mathbf{V}\nabla h + \frac{\partial h}{\partial t} - \frac{1}{\rho^{\circ}}\left(\mathbf{V}\nabla p + \frac{\partial p}{\partial t}\right) + N = 0 \\
(N = [(\mathbf{V}_d - \mathbf{V})\mathbf{f} + q]\,\rho_d/\rho - Q)
\end{gathered} \tag{3.1}$$

We might point out that the equations of continuity and motion agree with the corresponding equations of Rakhmatulin [1].

4. Equations (1.1), (1.2), and (2.2)-(2.4) also permit us to obtain the relations at second-order discontinuities.

Since a small discontinuity element may be considered plane, we shall limit ourselves to plane discontinuities. By properly choosing the position and constant velocity of the coordinate system, we make the discontinuity surface coincide (at a given instant) with the $x = 0$ plane. We denote the projections of $\mathbf{V}$ and $\mathbf{V}_d$ on the x-axis by V_n and V_{nd}, and the components parallel to the discontinuity surface by $\mathbf{V}_{\tau}$ and $\mathbf{V}_{\tau d}$. A jump of arbitrary magnitude ϕ at the discontinuity will be denoted in the present section by $[\phi] \equiv \phi_+ - \phi_-$, where the minus sign is associated with parameters for $x < 0$ and the plus sign for $x > 0$.

Let us consider a cylinder with bases of unit area lying in the planes $x = \pm\,\epsilon$ and the generator parallel to x, and apply relations (2.2)-(2.4) to it. Integrating, we eliminate the derivatives with respect to x from the volume integrals. Taking the limit as $\epsilon \to 0$, we take into account the finiteness of $\mathbf{f}$, $\mathbf{F}$, $\mathbf{F}_d$, q, Q, Q_d, and of the derivatives with respect to y, z, and t.

From (2.2) we have

$$[\rho V_n] = 0, \qquad [\rho_d V_{nd}] = 0 \tag{4.1}$$

These conditions permit us to introduce the fluxes $j = \rho V_n$ and $j_d = \rho_d V_{nd}$, which are continuous over the transition through the discontinuity. Similarly, (1.1), (1.2), (2.3), (2.4), and (4.1) give us the remaining relations at the discontinuity, which after several manipulations become

$$\begin{gathered} j\,[V_n] + j_d\,[V_{nd}] + [p] = 0, \qquad j\,[\mathbf{V}_\tau] = 0 \\ [V_{nd}^2] + 2\,[p]\,/\,\rho_d^\circ = 0, \qquad j_d[\mathbf{V}_{\tau d}] = 0 \\ j\,[{}^1/{}_2V_n^2 + h] = 0, \qquad j_d[e_d] = 0 \end{gathered} \tag{4.2}$$

For a given flow on one side of the discontinuity, (4.1) and (4.2) along with (2.1) and (2.5) together determine the flow on the other side.

Let us investigate the resulting expressions, neglecting the case when the gas is absent to at least one side of the discontinuity, i.e., assuming that $\rho_- > 0$ and $\rho_+ > 0$. In addition, since the case $\rho_{d-} = \rho_{d+} = 0$ is of no interest, we assume that $\rho_{d-} > 0$.

Five fundamentally distinct cases are possible.

Case 1. $j = j_d = 0$, i.e., there is no flow of the medium through the discontinuity.

By analysis of conditions (4.2) we have

$$[p] = 0, \qquad V_{n-} = V_{n+} = V_{nd-} = V_{nd+} = 0$$

The values of $[\mathbf{V}_\tau]$, $[\mathbf{V}_{\tau d}]$, $[\rho_d]$, $[e_d]$ and $[T]$ are arbitrary and determine, in accordance with (2.5) and (2.1), the jumps in ρ°, h, and ρ. We have a tangential discontinuity common to both media.

Case 2. $j = 0$, $j_d \neq 0$, i.e., a particle stream is present in the absence of a gas stream. From the first and third of conditions (4.2) and from the fact that $\rho_d/\rho_d^\circ < 1$ we find that

$$[p] = 0, \quad [V_{nd}] = 0$$

From this, (4.1), and the remaining equations of (4.2) we have

$$[\rho_d] = 0, \qquad [e_d] = 0, \qquad [\mathbf{V}_{\tau d}] = 0$$

The jumps in $\mathbf{V}_\tau$ and T are arbitrary and [by virtue of (2.5) and (2.1)] determine the jumps in ρ°, h, and ρ. In this case the tangential discontinuity in the gas is intersected by a continuous stream of particles.

Case 3. $j \neq 0$, $j_d = 0$, $\rho_{d-} > 0$, $\rho_{d+} > 0$, i.e., only the gas flows through the discontinuity, although particles are present everywhere. From this condition, using the third, first, and fifth equations of (4.2), the first condition of (4.1), and the second equation of (4.2) in that order, we have

$$V_{nd-} = V_{nd+} = 0, \qquad [p] = [h] = [\rho] = [V_n] = 0, \qquad [\mathbf{V}_\tau] = 0$$

By (2.5), the continuity of p, h, and ρ implies the continuity of T and ρ^0, as well as the continuity of ρ_d [by virtue of (2.1)]. The discontinuities in $\mathbf{V}_{\tau d}$ and e_d are arbitrary. We therefore have a continuous flow of gas through the surface of the tangential discontinuity in the 'continuous medium' of particles. The continuity of the gas parameters is assured by the continuity of the particle density.

A characteristic feature of the types of discontinuities just considered is the continuity of pressure.

Case 4. $j \neq 0,\ j_d = 0,\ \rho_{d-} > 0,\ \rho_{d+} = 0$; in this case there is a flow of gas through the boundary of the region containing the particles. The first and fifth equations of (4.2) now yield the two relations

$$[p] = j^2 \left\{ \frac{1}{\rho_-^\circ} \left(1 - \frac{\rho_{d-}}{\rho_d^\circ}\right)^{-1} - \frac{1}{\rho_+^\circ} \right\}, \qquad 2\,[h] = \left\{ \frac{1}{\rho_+^\circ} + \frac{1}{\rho_-^\circ} \left(1 - \frac{\rho_{d-}}{\rho_d^\circ}\right)^{-1} \right\} [p]$$

which together with expressions (2.5) for ρ^0 and h determine $[p]$, $[T]$, $[h]$, and $[\rho^0]$; moreover $\rho_+ = \rho_+^0$. As we see from the latter equation, the signs of $[p]$ and $[h]$ coincide. Further, on the basis of (4.1) and the first two equations of (4.2)

$$[V_n] = -\,[p]/j, \qquad \rho_+/\rho_- = V_{n-}/V_{n+}, \qquad [\mathbf{V}_\tau] = 0$$

Since no particles are present for $x > 0$ in this case, it follows that the meaning of the parameters V_{nd+}, $\mathbf{V}_{\tau d+}$, and e_{d+} is highly arbitrary. Nonetheless, they too can be determined. $[\mathbf{V}_{\tau d}]$ and $[e_d]$ are arbitrary and $V_{nd+}^2 = -(2/\rho_d^\circ)\,[p]$. Using the terminology of Kleiman[2], we call these 'combined' discontinuities.

Case 5. $j \neq 0$, $j_d \neq 0$. In this case the second, fourth, and sixth conditions of (4.2) yield

$$[\mathbf{V}_\tau] = 0, \qquad [\mathbf{V}_{\tau d}] = 0, \qquad [e_d] = 0$$

i.e., the indicated parameters are continuous.

With due regard for (2.1), the first and third equations of (4.2) may be written as

$$j^2 = \frac{[p]}{[v]} \left(\frac{2v_d^\circ}{v_{d-} + v_{d+}} - 1 \right), \qquad j_d^2 = -\frac{2v_d^\circ\,[p]}{[v_d^2]} \tag{4.3}$$

where v denotes the corresponding specific volumes. Since $2v_d^\circ < v_{d-} + v_{d+}$, (4.1)-(4.3) imply that the signs of $[V_n]$, $[V_{nd}]$, $[v]$ and $[v_d]$ coincide and are opposite to those of $[p]$, $[h]$, $[\rho]$ and $[\rho_d]$. The discontinuities under consideration are shock waves.

The equation of the Hugoniot curve is obtained from (4.2), (4.3), and (2.1) and has the form

$$[h] - \frac{v_-^\circ + v_+^\circ}{2}[p] - \left\{\frac{[p]\,v_d^\circ}{j_d\,(v_{d-} + v_{d+})}\right\}^2 \left(\frac{v_+^\circ}{v_{d+} - v_d^\circ} - \frac{v_-^\circ}{v_d - v_d^\circ}\right) = 0$$
$$(v_{d+} = \sqrt{v_{d-}^2 - 2j_d^{-2}\,v_d^\circ\,[p]}) \qquad (4.4)$$

Since h and v° are functions of p and T, it follows that for a known flow for $x < 0$ this equation gives the relationship between v_+° and p_+. The parameters for $x > 0$ are determined by solving (4.4) and the first equation of (4.3) simultaneously. After v_+ and v_{d+} have been determined, the values of V_{n+} and V_{nd+} are found from (4.1).

Gas flow through discontinuities of the two latter types is accompanied by changes in the thermodynamic parameters of the gas. In the case of the combined discontinuity this is due to the finiteness of the volume occupied by the particles, i.e., with the finiteness of $v_d^\circ = 1\ \rho_d^\circ/6$. This is also the reason why (4.4) differs from the Hugoniot curve in a pure gas. Since $\rho_d = \pi\,\rho_d^\circ/6$ for a densely packed particle distribution, it is always the case that $\rho_d < \rho_d^\circ$. In the case of moderate pressures in the flow of a gas containing solid or liquid particles we also have $\rho^\circ \ll \rho_d^\circ$. Under such circumstances the effect of the finiteness of v_d° is small, and in the first approximation we can set $v_d^\circ = 0$. Changes in the parameters of the gas in combined discontinuities and of the particle stream in shock waves do not occur, and the parameters of the gas in the shock wave undergo the same changes as those of a pure gas. Rudinger [5] computed the forward jumps in packing density employing this approximation. Taking into account the smallness of the jumps in the thermodynamic parameters of the gas in combined discontinuities, we obtain

$$T_-[s] \approx \frac{v_d^\circ v_-^\circ}{2\,(v_{d-} - v_d^\circ)}[p]$$

where s is the specific entropy of the gas. From this expression it is easy to determine the direction of possible changes in the parameters.

5. Initial and boundary conditions are necessary in the solution of many of the problems. The formulation of the initial conditions for the entire medium and of the boundary conditions imposed on the parameters of the gas does not differ from that for the flow of an ideal continuous medium. Specifically, the normal component of the velocity $\mathbf{V}$ vanishes on hard surfaces. We do not need such boundary conditions to solve the equations describing the motion of particles, so that in the general case $\mathbf{V}_d$ has a component normal to the hard surface, and nonpermeation by the particles is guaranteed by their reflection according to laws determined by the nature of the particles and surfaces. The presence of reflected flows complicates the flow picture, making it necessary to consider multiple-velocity rather than just two-velocity media. The problem is simplified where it can be assumed that the particles are absorbed by the hard surfaces, as, for example, in the case of liquid droplets.

6. We begin our investigation of various types of flows with the case of one-dimensional stationary flow with plane, cylindrical, or spherical symmetry. The distance from the

corresponding plane, axis, or center is denoted by r. Since the velocities and forces in this case are directed along r, it follows that (1.1), (1.2), and (1.3) become

$$\rho \frac{\partial V}{\partial r} + V \frac{\partial \rho}{\partial r} + \frac{\partial \rho}{\partial t} + \frac{\nu \rho V}{r} = 0 \tag{6.1}$$

$$\rho_d \frac{\partial V_d}{\partial r} + V_d \frac{\partial \rho_d}{\partial r} + \frac{\partial \rho_d}{\partial t} + \frac{\nu \rho_d V_d}{r} = 0 \tag{6.2}$$

$$V \frac{\partial V}{\partial r} + \frac{\partial V}{\partial t} + \frac{1}{\rho^\circ} \frac{\partial p}{\partial r} + \frac{\rho_d}{\rho} f - F = 0 \tag{6.3}$$

$$V_d \frac{\partial V_d}{\partial r} + \frac{\partial V_d}{\partial t} + \frac{1}{\rho_d^\circ} \frac{\partial p}{\partial r} - f - F_d = 0 \tag{6.4}$$

$$V \frac{\partial h}{\partial r} + \frac{\partial h}{\partial t} - \frac{1}{\rho^\circ}\left(V \frac{\partial p}{\partial r} + \frac{\partial p}{\partial t}\right) + N = 0 \tag{6.5}$$

$$V_d \frac{\partial e_d}{\partial r} + \frac{\partial e_d}{\partial t} - q - Q_d = 0 \tag{6.6}$$

where $\nu = 0$, 1, and 2, respectively, in the plane, cylindrical and spherical cases.

Let us consider the problem of the characteristics of system (6.1)-(6.6). First of all, the form of equations (6.5) and (6.6) implies that the characteristics are the trajectories of the gas and particles. If the total derivatives with respect to t along the characteristics are denoted by a prime, the equations of the trajectories and the relations along them become

$$r' - V = 0, \qquad h' - (1 / \rho^\circ)\, p' + N = 0 \tag{6.7}$$

$$r' - V_d = 0, \qquad e_d' - q - Q_d = 0 \tag{6.8}$$

The remaining equations contain derivatives of p, V, V_d, ρ, and ρ_d. Derivatives of ρ appear in the first equation only and can be eliminated with the aid of (2.1), (2.5), and (6.5). As a result, instead of (6.1) we obtain

$$\rho^\circ a^2 \frac{\partial V}{\partial r} + V \frac{\partial p}{\partial r} + \frac{\partial p}{\partial t} - V \frac{\rho^{\circ 2} a^2}{\rho_d^\circ \rho} \frac{\partial \rho_d}{\partial r} - \frac{\rho^{\circ 2} a^2}{\rho_d^\circ \rho} \frac{\partial \rho_d}{\partial t} - \frac{a^2 \rho_T^\circ}{h_T} N + \frac{\nu \rho^\circ a^2 V}{r} = 0 \tag{6.9}$$

where a is the speed of sound in the gas,

$$a^{-2} = \rho_p^\circ + \frac{\rho_T^\circ}{h_T}\left(\frac{1}{\rho^\circ} - h_p\right) \tag{6.10}$$

and

$$\rho_p^\circ = \left(\frac{\partial \rho^\circ}{\partial p}\right)_T, \qquad \rho_T^\circ = \left(\frac{\partial \rho^\circ}{\partial T}\right)_p, \qquad h_p = \left(\frac{\partial h}{\partial p}\right)_T, \qquad h_T = \left(\frac{\partial h}{\partial T}\right)_p$$

Adding to (6.2)-(6.4) and (6.9) the expression for the increment in p,

$$\frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial t} dt = dp$$

and the analogous relations for dV, dV_d, and $d\rho_d$, we obtain a system of linear algebraic equations for determining the partial derivatives of p, V, V_d, and ρ_d. The condition of ambiguity of the solution of this system (equality to zero of the corresponding determinants) leads to the equations

$$(V_d - r')^2 [a^2 - (V - r')^2] + \frac{\rho^{\circ 2}\rho_d a^2}{\rho_d^{\circ 2}\rho}(V - r')^2 = 0 \tag{6.11}$$

$$\begin{aligned}(V_d - r')^2 \Big\{V' - \frac{V - r'}{\rho^\circ a^2} p' + (V - r')\left(\frac{\rho_T{}^\circ N}{\rho^\circ h_T} - \frac{\nu V}{r}\right) + \\ + \frac{\rho_d}{\rho} f - F + \frac{\rho^\circ \rho_d (V - r')}{\rho_d{}^\circ \rho (V_d - r')}\Big[\frac{V - r'}{V_d - r'} V_d' + (V_d - V) \times \\ \times (\ln \rho_d)' - \frac{V - r'}{V_d - r'}\left(f + F_d\right) - (V - r')\frac{\nu V_d}{r}\Big]\Big\} = 0\end{aligned} \tag{6.12}$$

Equations (6.11) and (6.12) determine the characteristics different from the trajectories; (6.11) gives the directions of the characteristics and (6.12) the changes in the parameters along them.

To investigate the form of the roots of equation (6.11) we first consider the point in the flow where $V = V_d$. Here (6.11) becomes a product of two cofactors,

$$(V_d - r')^2 [b^2 - (V_d - r')^2] = 0, \qquad b^2 = a^2\left(1 + \frac{\rho^{\circ 2}\rho_d}{\rho_d{}^{\circ 2}\rho}\right) > a^2$$

and therefore has the two distinct $r_{1,2}' = V_d \pm b$ and the multiple root $r' = V_d$. In the second instance (6.12) vanishes by virtue of the first factor, and therefore cannot lead to any additional conditions. If $V \neq V_d$, the solution of (6.11) cannot be obtained in explicit form. However, in view of the smallness of $v_d^\circ = 1/\rho_d^\circ$ it is not difficult to find the expansions of the roots in powers of v_d°. As a result we have

$$\begin{aligned} r_{1,2}' &= V \pm a\left[1 + \frac{\rho^{\circ 2}\rho_d a^2}{2\rho_d{}^{\circ 2}\rho (V_d - V \mp a)^2}\right] + O(v_d{}^{\circ 4}) \\ r_{3,4}' &= V_d \pm \frac{\rho^\circ a (V - V_d)}{\rho_d{}^\circ}\left\{\frac{\rho}{\rho_d}[(V - V_d)^2 - a^2]\right\}^{-1/2} + O(v_d{}^{\circ 2}) \end{aligned} \tag{6.13}$$

The first two roots are real and for $V = V_d$, coincide with $V \pm b$ to within infinitesimals of higher order. The third and fourth roots for $V \neq V_d$ are real only for $|V - V_d| > a$, i.e., if the relative velocity of the particles exceeds the speed of sound in the gas. This case exceeds the confines of the present theory, however. Thus, in addition to the trajectories the equations of one-dimensional nonstationary motion always have two families of real characteristics along which equations (6.13) and (6.12) are fulfilled. To within an accuracy of $o(v_d^\circ)$ equation (6.12) may be written as

$$V' \pm \frac{1}{\rho^\circ a} p' \mp a\left(\frac{\rho_T{}^\circ N}{\rho^\circ h_T} - \frac{\nu V}{r}\right) + \frac{\rho_d}{\rho} f - F + \frac{\rho^\circ \rho_d a}{\rho_d{}^\circ \rho (V_d - V \mp a)^2} \times$$

$$\times\left[aV_d' \mp (V_d - V)(V_d - V \mp a)(\ln \rho_d)' - a(f + F_d) - a(V_d - V \mp a)\frac{\nu V_d}{r}\right] = 0$$

where the upper (lower) sign corresponds to the characteristics of the first (second) family.

7. Another interesting class of flows that depends on just two variables consists of plane and axisymmetrical stationary flows. Let x, y be rectangular coordinates; in the axisymmetrical case the x-axis is directed along the axis of symmetry in the left-to-right direction. Projections of the forces and velocities are denoted by the appropriate subscripts. Equations (1.1), (1.2), and (3.1) then become

$$\rho\frac{\partial V_x}{\partial x} + \rho\frac{\partial V_y}{\partial y} + V_x\frac{\partial \rho}{\partial x} + V_y\frac{\partial \rho}{\partial y} + \frac{\nu\rho V_y}{y} = 0 \tag{7.1}$$

$$\rho_d\frac{\partial V_{xd}}{\partial x} + \rho_d\frac{\partial V_{yd}}{\partial y} + V_{xd}\frac{\partial \rho_d}{\partial x} + V_{yd}\frac{\partial \rho_d}{\partial y} + \frac{\nu\rho_d V_{yd}}{y} = 0 \tag{7.2}$$

$$V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + \frac{1}{\rho^\circ}\frac{\partial p}{\partial x} + \frac{\rho_d}{\rho}f_x - F_x = 0 \tag{7.3}$$

$$V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + \frac{1}{\rho^\circ}\frac{\partial p}{\partial y} + \frac{\rho_d}{\rho}f_y - F_y = 0 \tag{7.4}$$

$$V_{xd}\frac{\partial V_{xd}}{\partial x} + V_{yd}\frac{\partial V_{xd}}{\partial y} + \frac{1}{\rho_d^\circ}\frac{\partial p}{\partial x} - f_x - F_{xd} = 0 \tag{7.5}$$

$$V_{xd}\frac{\partial V_{yd}}{\partial x} + V_{yd}\frac{\partial V_{yd}}{\partial y} + \frac{1}{\rho_d^\circ}\frac{\partial p}{\partial y} - f_y - F_{yd} = 0 \tag{7.6}$$

$$V_x\frac{\partial h}{\partial x} + V_y\frac{\partial h}{\partial y} - \frac{V_x}{\rho^\circ}\frac{\partial p}{\partial x} - \frac{V_y}{\rho^\circ}\frac{\partial p}{\partial y} + N = 0 \tag{7.7}$$

$$V_{xd}\frac{\partial e_d}{\partial x} + V_{yd}\frac{\partial e_d}{\partial y} - q - Q_d = 0 \tag{7.8}$$

where $\nu = 0$ and 1, respectively, for the plane and axisymmetrical cases.

Let us find the characteristics of system (7.1)-(7.8). The form of equations (7.7) and (7.8) shows first of all that the gas and particle streamlines are characteristics. Denoting the total derivatives with respect to x along the characteristics with a prime, we find that along the gas streamlines

$$\begin{gathered} V_x y' - V_y = 0, \qquad V_x h' - (V_x / \rho^\circ)\, p' + N = 0 \\ V_x\,(h + {}^1/_2 V^2)' + (\mathbf{f}\mathbf{V}_d + q)\,\rho_d / \rho - Q - \mathbf{F}\mathbf{V} = 0 \end{gathered} \tag{7.9}$$

Similarly, along the particle streamlines

$$\begin{gathered} V_{xd} y' - V_{yd} = 0, \qquad V_{xd} e_d' - q - Q_d = 0 \\ V_{xd}\,({}^1/_2 V_d^2 + p / \rho_d^\circ)' - \mathbf{f}\mathbf{V}_d - \mathbf{F}_d\mathbf{V}_d = 0 \end{gathered} \tag{7.10}$$

The latter equation is the result of adding (7.5) and (7.6) multiplied by V_{xd} and V_{yd}.

It will be convenient for us to exclude the derivatives of ρ at this point, just as we did in the case of nonstationary flows. As a result, in place of (7.1) we have

$$(a^2 - V_x^2)\frac{\partial V_x}{\partial x} - V_xV_y\frac{\partial V_x}{\partial y} - V_xV_y\frac{\partial V_y}{\partial x} + (a^2 - V_y^2)\frac{\partial V_y}{\partial y} - \frac{\rho^\circ a^2 V_x}{\rho_d^\circ\rho}\frac{\partial\rho_d}{\partial x} -$$

$$-\frac{\rho^\circ a^2 V_y}{\rho_d^\circ\rho}\frac{\partial\rho_d}{\partial y} + \frac{\nu a^2 V_y}{y} - K = 0 \qquad \left(K = \left(\frac{\rho_d}{\rho}\mathbf{f} - \mathbf{F}\right)\mathbf{V} + \frac{\rho_T{}^\circ a^2}{\rho^\circ h_T}N\right) \tag{7.11}$$

Equations (7.2)-(7.6) and (7.11) together with the expression for the increment in p

$$\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = dp$$

and similar expressions for dV_x, dV_y, dV_{xd}, dV_{yd} and $d\rho_d$ form a system of algebraic equations from which we can determine all partial derivatives of p, V_x, V_y, V_{xd}, V_{yd} and ρ_d. The equations of the characteristics may once again be obtained from the condition of ambiguity (non-single valuedness) of the solution of this system.

The values of y' on the characteristics satisfy the condition

$$AB\left\{B^2\left[(1+y'^2)\,a^2 - A^2\right] + \frac{\rho_d\rho^{\circ 2}a^2}{\rho_d{}^{\circ 2}\rho}(1+y'^2)\,A^2\right\} = 0$$

$$(A = V_y - y'V_x,\ B = V_{yd} - y'V_{xd})$$

Thus, in addition to the streamlines, the lines whose directions satisfy the equation

$$B^2\left[(1+y'^2)\,a^2 - A^2\right] + \frac{\rho_d\rho^{\circ 2}a^2}{\rho_d{}^{\circ 2}\rho}(1+y'^2)\,A^2 = 0 \tag{7.12}$$

are also characteristics.

As regards the relations along the characteristics, the corresponding conditions lead to closing equations from (7.9) and (7.10) and to the equation

$$\frac{a^2y' + AV_x}{\rho^\circ}p' + a^2V_yV_x' + (a^2y'^2 - A^2)\,V_xV_y' + Aa^2\left(\frac{\nu V_y}{y} - \frac{K}{a^2}\right) -$$

$$-(a^2y' + AV_x)\,(M + y'L) + \frac{\rho^\circ\rho_d a^2A^2}{\rho_d{}^\circ\rho B^2}\left[V_{yd}^2\left(\frac{V_{xd}}{V_{yd}}\right)' + \frac{B\,(V_yV_{xd} - V_xV_{yd})\,\rho_d'}{A\rho_d} + \right.$$

$$\left. + f_y + F_{yd} - y'\,(f_x + F_{xd}) + \frac{\nu V_{yd}B}{y}\right] + \frac{\rho^{\circ 2}\rho_d a^2A^2}{\rho_d{}^{\circ 2}\rho B^2}\left[\frac{y'}{\rho^\circ}p' + \right. \tag{7.13}$$

$$\left. + (1+y'^2)\,(V_xV_y' - L)\right] = 0 \qquad \left(M = F_x - \frac{\rho_d}{\rho}f_x,\ L = F_y - \frac{\rho_d}{\rho}f_y\right)$$

which is fulfilled along the characteristics which are not streamlines.

To analyze the solutions of equation (7.12) we make use of expansions of its roots in powers of $\overset{\circ}{v}_d$. Such an expansion yields

$$y_{1,2}' = y_{\infty 1,2}' \pm \frac{\rho^{\circ 2}\rho_d\,(V_y - y_{\infty 1,2}'V_x)^4}{2\rho_d^{\circ 2}\rho a\,\sqrt{V^2 - a^2}\,(V_{yd} - y_{\infty 1,2}'V_{xd})^2} + O\,(v_d^{\circ 4}) \tag{7.14}$$

$$y_{3,4}' = \frac{V_{yd}}{V_{xd}} \pm \frac{\rho^\circ a V V_d \sin\delta}{\rho_d^\circ V_{xd}^2}\left[\frac{\rho}{\rho_d}\,(V^2\sin^2\delta - a^2)\right]^{-1/2} + O\,(v_d^{\circ 2})$$
$$\left(y_{\infty 1,2}' = \frac{V_xV_y \pm a\,\sqrt{V^2 - a^2}}{V_x^2 - a^2}\right) \tag{7.15}$$

(δ is the angle between $\mathbf{V}$ and $\mathbf{V}_d$).

(7.15) implies that the roots $y_{3,4}'$ are real only if the component of the gas velocity normal to $\mathbf{V}_d$ exceeds the speed of sound in the gas. However, as with nonstationary flow, the present theory is not valid for this case.

Roots (7.14) are real for $V > a$ and imaginary for $V < a$. In the first instance they correspond to the characteristics of the first and second families of conventional gas dynamics. For $V_x \to a$, as $y_\infty' \to \infty$, in place of (7.14) one must make use of the expansion

$$x_{1,2}' = x_{\infty 1,2}' \mp \frac{\rho^{\circ 2}\rho_d\,(x_{\infty 1,2}'V_y - V_x)^4}{2\rho_d^{\circ 2}\rho a\,\sqrt{V^2 - a^2}\,(x_{\infty 1,2}'V_{yd} - V_{xd})^2} + O\,(v_d^{\circ 4})$$
$$\left(x_{\infty 1,2} = \frac{V_x^2 - a^2}{V_xV_y \pm a\,\sqrt{V^2 - a^2}}\right) \tag{7.16}$$

where the prime denotes derivatives with respect to y. In (7.14), (7.16) and below the upper (lower) sign and the subscript 1 (2) correspond to the characteristics of the first (second) family.

The boundary of the region of hyperbolic behaviour found from (7.14) and (7.16) is not exact, since these expansions are not valid for $V = a$. To determine the boundary we make use of the fact that within the region of validity of the present theory the component of $\mathbf{V}_d$ normal to $\mathbf{V}$, equal to $V_d \sin\delta$, is small. Expanding the roots of (7.12) in $\Delta = V_d \sin\delta$, we have

$$\tan\alpha_{1,2} = \pm\frac{b}{\sqrt{V^2 - b^2}} + \frac{\rho^{\circ 2}\rho_d a^2 V^7 V_d \sin\delta}{\rho_d^{\circ 2}\rho b^2\,(V^2 - b^2)\,(\mathbf{V}\mathbf{V}_d)^3} + O\,(\Delta^2)$$

$$\cot\alpha_{1,2} = \pm\frac{\sqrt{V^2 - b^2}}{b} - \frac{\rho^{\circ 2}\rho_d a^2 V^7 V_d \sin\delta}{\rho_d^{\circ 2}\rho b^4\,(\mathbf{V}\mathbf{V}_d)^3} + O\,(\Delta^2), \qquad b^2 = a^2\left[1 + \frac{\rho^{\circ 2}\rho_d V^4}{\rho_d^{\circ 2}\rho\,(\mathbf{V}\mathbf{V}_d)^2}\right]$$

where α is the angle between the characteristic and the gas streamline. Thus, with $V \geqslant b$

there are two families of real characteristics in addition to the streamlines. In contrast to the case of a pure gas, they are not situated symmetrically with respect to the streamlines, but are rotated in the direction of the particle streamlines. To within $o\,(v_d^{\circ})$ expression (7.13) has the form

$$\mp \frac{\sqrt{V^2-a^2}}{\rho^\circ a}\,p' + V_y^2\left(\frac{V_x}{V_y}\right)' + A_{1,2}\left(\frac{\nu V_y}{y}-\frac{K}{a^2}\right) \pm \frac{\sqrt{V^2-a^2}}{a}\,(M + y_{\infty 1,2}'L) +$$
$$+\frac{\rho^\circ\rho_d A_{1,2}^2}{\rho_d{}^\circ\rho B_{1,2}^2}\left[V_{yd}^2\left(\frac{V_{xd}}{V_{yd}}\right)' + \frac{B_{1,2}\,(V_yV_{xd}-V_xV_{yd})}{A_{1,2}\rho_d}\,\rho_d' + f_y + F_{yd} -\right.$$
$$\left.- y_{\infty 1,2}'\,(f_x+F_{xd}) + \frac{\nu V_{yd}B_{1,2}}{y}\right] = 0 \qquad (A_i = V_y - y_{\infty i}'V_x,\quad B_i = V_{yd} - y_{\infty i}'V_{xd})$$

on the characteristics of the first and second families.

8. We conclude our discussion of various classes of flows with an investigation of quasi-one-dimensional stationary flow. With the assumptions customary in this case, we find that (2.2)-(2.4), (1.1), and (1.2) yield

$$\rho VS = c, \qquad \rho_d V_d S = c_d$$
$$({}^1/{}_2V_d^2 + p\,/\,\rho_d{}^\circ)' - f - F_d = 0, \qquad V_d e_d' - q - Q_d = 0$$
$$\rho_d V_d V_d' + \rho V V' + p' - \rho F - \rho_d F_d = 0$$
$$[\rho VS\,({}^1/{}_2V^2 + h) + \rho_d V_d S\,({}^1/{}_2V_d^2 + e_d + p\,/\,\rho_d{}^\circ)]' -$$
$$- \rho\,(Q + VF) - \rho_d\,(Q_d + V_dF_d) = 0$$

where the prime denotes derivatives with respect to distance measured along the axis of the channel, S is the cross-sectional area, c and c_d are constants. This system generalizes equations usually employed to compute quasi-one-dimensional flows [6-9].

In the absence of external forces and heat fluxes, the last equation may be integrated to yield

$$\rho VS\,({}^1/{}_2V^2 + h) + \rho_d V_d S\,({}^1/{}_2V_d^2 + e_d + p\,/\,\rho_d{}^\circ) = \text{const}$$

Integration of the penultimate equation with S = const gives us

$$\rho V^2 + \rho_d V_d^2 + p = \text{const}$$

In this case the foregoing equations become exact.

9. To further our understanding of some of the peculiarities of two-velocity flows, let us consider the limiting process whereby two-velocity flow becomes one-velocity flow. Let ϕ^1 in (1.3) increase. By virtue of the finiteness of the force **f** we then have $\mathbf{V}_d \to \mathbf{V}$. For $\phi^1 = \infty$ the difference in velocities **V** and $\mathbf{V}_d$ vanishes, and on eliminating **f** the flow equations become

$$\nabla\rho_\Sigma \mathbf{V} + \frac{\partial\rho_\Sigma}{\partial t} = 0, \qquad \mathbf{V}\nabla\frac{\rho}{\rho_\Sigma} + \frac{\partial}{\partial t}\frac{\rho}{\rho_\Sigma} = 0$$

$$\rho_\Sigma(\mathbf{V}\nabla)\mathbf{V} + \rho_\Sigma\frac{\partial\mathbf{V}}{\partial t} + \nabla p - \rho_\Sigma\mathbf{F}_\Sigma = 0$$

$$\mathbf{V}\nabla h_\Sigma + \frac{\partial h_\Sigma}{\partial t} - \frac{1}{\rho_\Sigma}\left(\mathbf{V}\nabla p + \frac{\partial p}{\partial t}\right) - Q_\Sigma = 0, \qquad \mathbf{V}\nabla e_d + \frac{\partial e_d}{\partial t} - q - Q_d = 0$$

$$\rho_\Sigma = \rho + \rho_d = \rho^\circ\left(\frac{\rho}{\rho_\Sigma} + \frac{\rho_d}{\rho_\Sigma}\frac{\rho^\circ}{\rho_d^\circ}\right)^{-1}, \qquad h_\Sigma = \frac{\rho}{\rho_\Sigma}e + \frac{\rho_d}{\rho_\Sigma}e_d + \frac{p}{\rho_\Sigma}$$

$$\mathbf{F}_\Sigma = \frac{\rho}{\rho_\Sigma}\mathbf{F} + \frac{\rho_d}{\rho_\Sigma}\mathbf{F}_d, \qquad Q_\Sigma = \frac{\rho}{\rho_\Sigma}Q + \frac{\rho_d}{\rho_\Sigma}Q_d$$

Thus, in the limiting case we have a flow of a one-velocity medium of density ρ_Σ, enthalpy h_Σ, non-equilibrium parameter e_d, and unchanging relative concentrations of the gas and particles in the fluid element. The directions of the characteristics in this case [10] are determined by the speed of sound a_Σ frozen relative to e_d; a_Σ may be computed from (6.10), where ρ^0 is replaced by ρ_Σ and h by h_Σ. The derivatives of $\rho_{\Sigma T}$, $\rho_{\Sigma p}$, $h_{\Sigma T}$ and $h_{\Sigma p}$ are found for a fixed fluid element with a constant e_d.

After some manipulation we obtain

$$a_\Sigma^2 = a^2\left[1 + \frac{\rho_d}{\rho^\circ} - \frac{\rho_d}{\rho_d^\circ}\left(2 + \frac{\rho_d}{\rho^\circ}\right) + \frac{\rho_d^2}{\rho_d^{\circ 2}}\right]^{-1}$$

For small $v_d^* = 1/\rho_d^*$ we have $a > a_\Sigma$. Recalling that the directions of the two-velocity flow are determined by the speed a to within $o(v_d^*)$, we readily note the similarity between the two-velocity and limiting one-velocity flows on the one hand, and non-equilibrium and equilibrium flows on the other. This similarity is evidenced by the very occurrence of the limiting transition as $\phi^1 \to \infty$. Consequently, the same difficulties may be encountered in the computation of two-velocity flows as in the case of non-equilibrium flows [11, 12].

10. In using the model of a two-velocity medium it is necessary to bear in mind the assumptions on which it is based. This applies first of all to the condition $\rho_d < \pi\rho_d/6$, whose violation in any flow region indicates that the model is inapplicable in the given instance. The conditions of applicability of the theory are likewise violated when $\rho_d \ll \rho$, i.e., when the number of particles is small. In this case, however, it is possible to disregard the effect of the particles on gas flow, and then to determine their motion from the known flow field using (1.1) and (1.2), where the first two terms are replaced by their total derivatives with respect to time. Such an approach has been developed in a number of studies on the flow of a medium containing water droplets around airfoils and engine inlets [13, 14].

The authors are grateful to G.M. Bam-Zelikovich and G.G. Chenyi for their useful comments.

BIBLIOGRAPHY

1. Rakhmatulin, Kh. A., Osnovy gazodinamiki vzaimopronikaiushchikh dvizhenii szhimaemykh sred (Principles of the gas dynamics of interpenetrating motions of compressible media). *Prikladnaia matematika i mekhanika*, Vol. 20, No. 2, 1956.

2. Kleiman, Ia.Z., O rasprostranenii sil'nykh razryvov v mnogokomponentnoi srede (Propagation of strong discontinuities in a multi-component medium). *Prikladnaia matematika i mekhanika*, Vol. 22, No. 2, 1958.

3. Kliegel, J.R. and Nickerson, G.R., Flow of gas-particle mixtures in axially symmetric nozzles, ARS, Report 1713-61, 1961.

4. Kochin, N.E., Kibel', I.A. and Roze, N.V., *Teoreticheskaia gidromekhanika (Theoretical Hydromechanics)*, Vol. II. Fizmatgiz, 1963, pp. 18-20.

5. Rudinger, G., Some properties of shock relaxation in gas flows carrying small particles. *Phys. Fluids*, Vol. 7, No. 5, 1964.

6. Gilbert, M., Davis, L. and Altman, D., Velocity lag of particles in linearly accelerated combustion gases. *Jet Propulsion*, Vol. 25, No. 1, 1955.

7. Soo, S.L., Gas dynamic processes involving suspended solids. *A.I. Ch.E.J.*, Vol. 7, No. 3, 1961.

8. Kobzar', A.I., Yantovskii, E.I. and Tolmach, I.M., Techenie dvufaznoi smesi v kanale peremennogo secheniia (Flow of a two-phase mixture in a channel of variable cross section). *Izv. Akad. Nauk SSSR, Energetika i transport*, No. 4, 1964.

9. Duban, P. and Nicolas, J., Influence de la presence de particules solides sur l'écoulement dans une tuyère. *Rech. aeronaut.*, No. 92, 1963.

10. Broer, L.J.F., Characteristics of equations of motion of reacting gas. *J. Fluid Mech.*, Vol. 4, Part 3, 1958.

11. Katskova, O.N. and Kraiko, A.N., Raschet ploskikh i osesimmetrichnykh sverkhzvukovykh techenii pri nalichii neobratimykh protsessov (Computation of plane and axisymmetrical supersonic flows in the presence of irreversible processes). *Tr. Vychisl. tsentra Akad. Nauk SSSR*, 1964.

12. Galiun, N.S. and Kraiko, A.N., K raschetu neravnovesnykh techenii (Analysis of non-equilibrium flows). *Izv. Akad. Nauk SSSR, Mekhanika i mashinostroenie*, No. 6, 1964.

13. Serafini, J.S., Impingement of water droplets on wedges and double-wedge airfoils at supersonic speeds. *NASA*, Report 1159, 1954.

14. Brun, R.J., Cloud-droplet ingestion in engine inlets with inlet velocity ratios of 1.0 and 0.7 *NASA*, Report 1317, 1957.

Translated by A.Y.